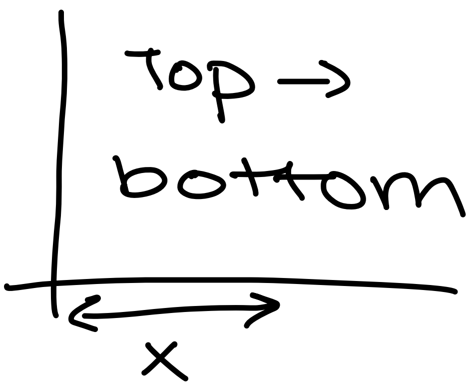
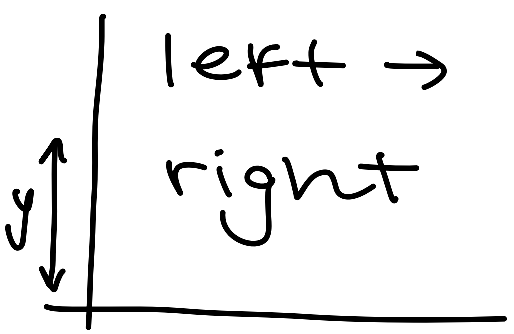
15.2 Double Integrals over General Regions

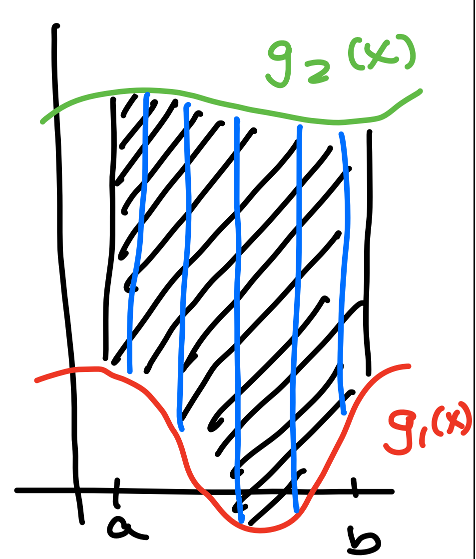
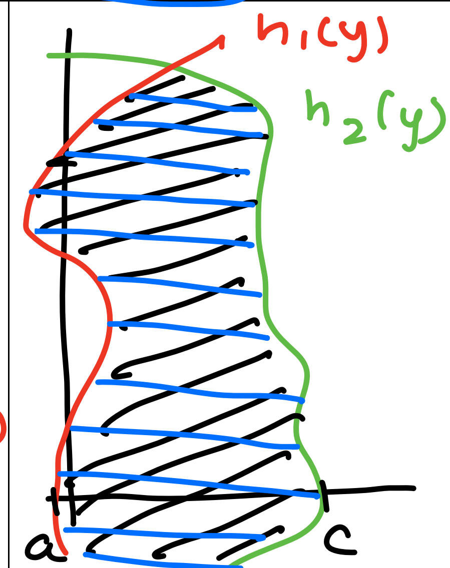
The surface $z = x + 3y^2$ over the rectangular region $R = [0,1] \times [0,3]$

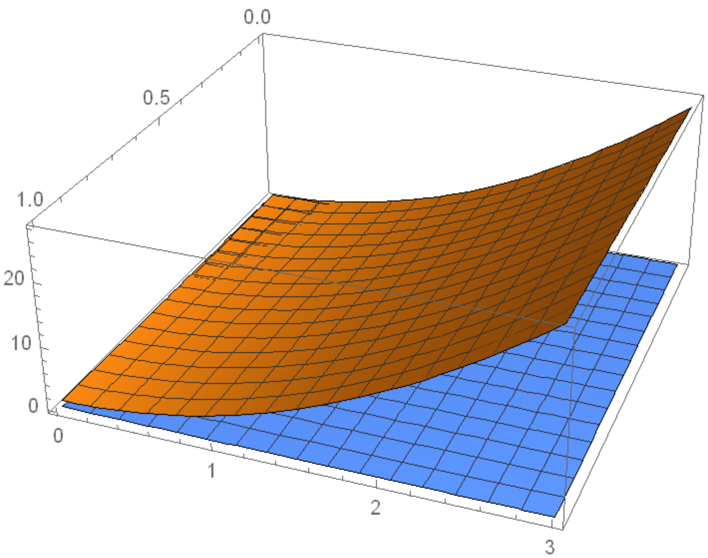
In 15.2, we discuss regions, *R*, other than rectangles.

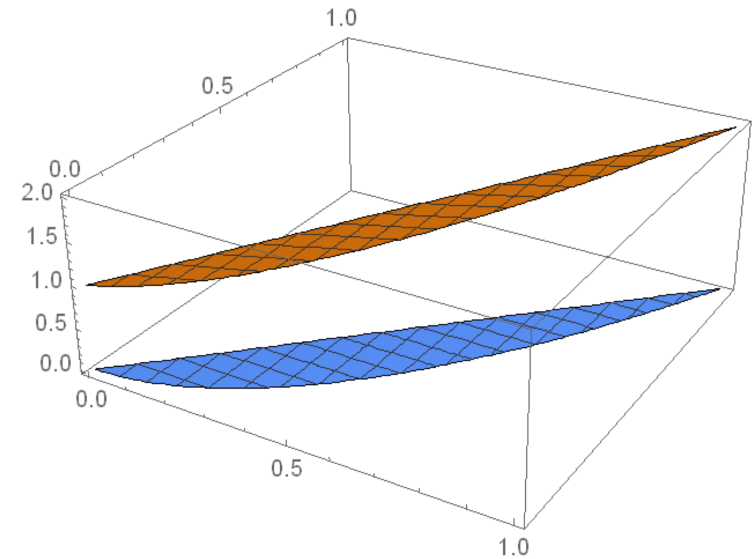
$$\iint_R x + 3y^2 \, dA$$





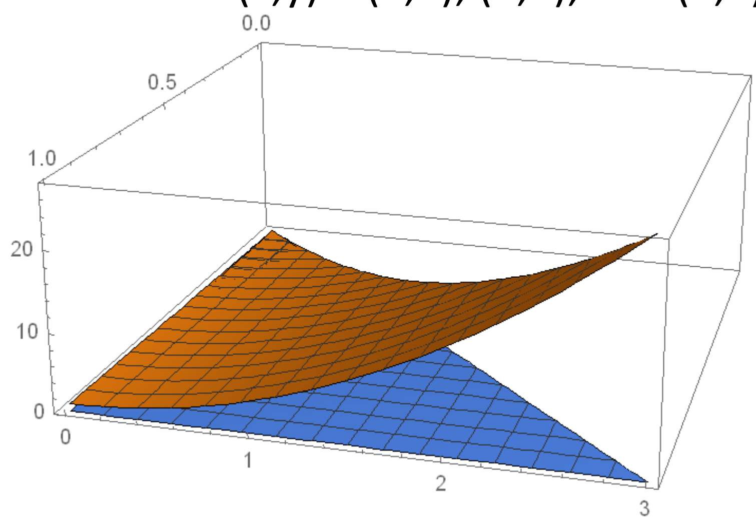
Type 1 Regions (Top/Bot)	Type 2 Regions (Left/Right)
	
Given *x* in the range, $a \le x \le b$, we have $g_1(x) \le y \le g_2(x)$	Given *y* in the range, $c \le y \le d$, we have $h_1(y) \le x \le h_2(y)$
$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx$	$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right) dy$

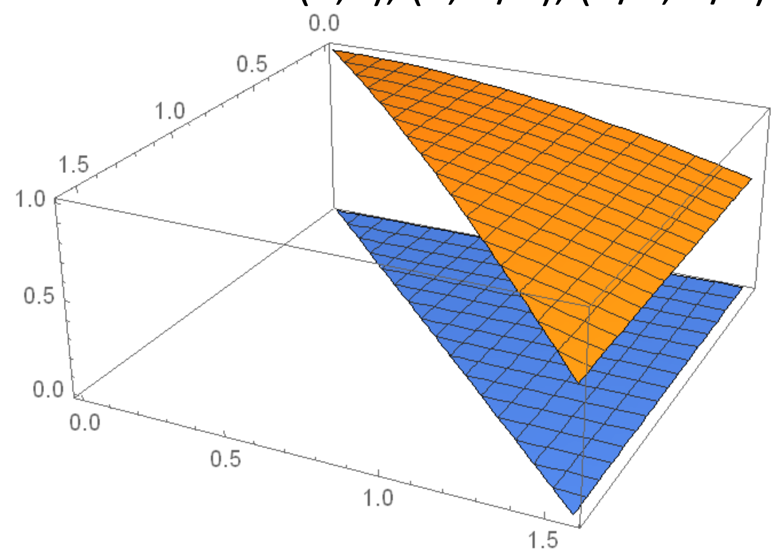




The surface $z = x + 3y^2$ over the triangular region with corners $(x,y) = (0,0)$, $(1,0)$, and $(1,3)$.

The surface $z = \sin(y)/y$ over the triangular region with corners at $(0,0)$, $(0, \pi/2)$, $(\pi/2, \pi/2)$.





The surface $z = x + 1$ over the region bounded by $y = x$ and $y = x^2$.

Example:

Draw the region, *R*, bounded by

$y = x^2, y = 2x + 3$.

Then set up the double integral

$$\iint_R f(x,y)\,dA$$

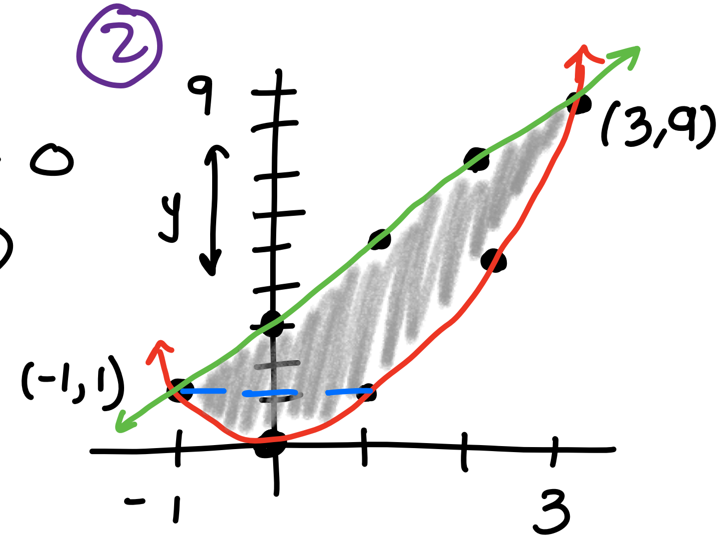
(Try it in both orders)

$x^2 = 2x + 3$

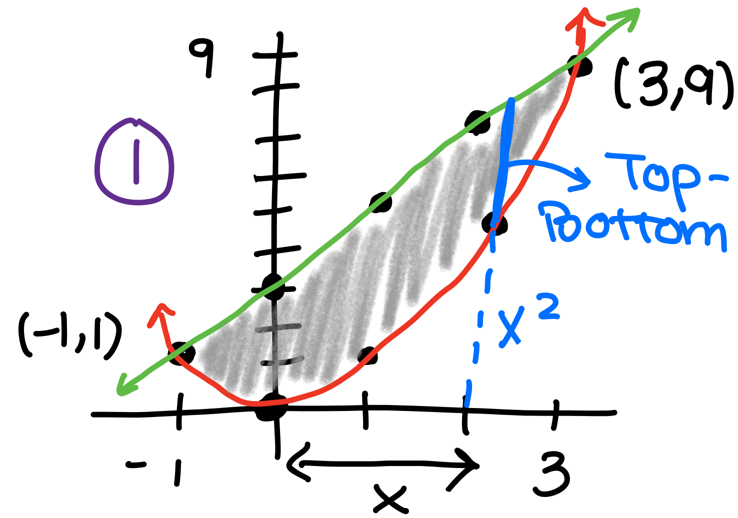
$x^2 - 2x - 3 = 0$

$(x+1)(x-3)$

$x = 3, -1$



1 $$\int_{-1}^{3}\left(\int_{x^2}^{2x+3} \overbrace{f(x,y)}^{\text{Surface in 3D}}\,dy\right)dx$$



2

$x = \sqrt{y}$ $x = -\sqrt{y}$ | $1 \le y \le 9$

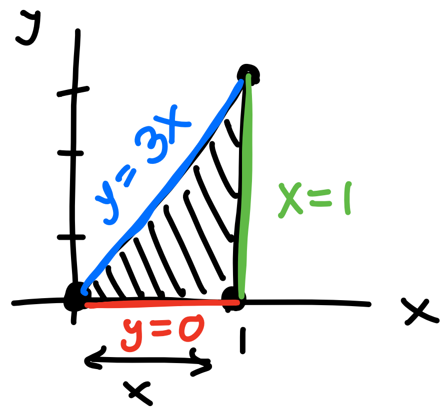
$0 \le y \le 1$ | $y - \frac{3}{2} \le x \le \sqrt{y}$

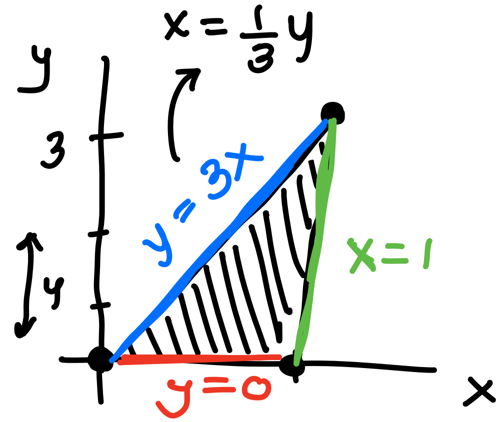
$-\sqrt{y} \le x \le \sqrt{y}$

$$\int_0^1\left(\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\right)dy + \int_1^9\left(\int_{y-3}^{\sqrt{y}} f(x,y)\,dx\right)dy$$

Visual: https://www.math3d.org/2fos5PZJ

Example:
Let D be the triangular region in the xy-plane with corners (0,0), (1,0), (1,3).





Evaluate $\iint_D x + 3y^2 dA$

NO y → Top

NO x → Right

$$\int_0^1 \left(\int_0^{3x} x+3y^2 \, dy \right) dx = \int_0^3 \left(\int_{1/3 y}^{1} x+3y^2 \, dx \right) dy$$

Bottom

Left

$$\int_0^1 \left(xy + y^3 \Big|_0^{3x} \right) dx$$

plug into y b/c dy

Should be no y here

$$\int_0^1 \left(3x^2 + 27x^3 \right) dx = x^3 + \frac{27}{4} x^4 \Big|_0^1 = \boxed{\frac{31}{4}}$$

Setting up a problem given in "words":

1. Find integrand

Solve for "z" anywhere you see it.
If there are two z's, then set up two double integrals (subtract at end).

2. Region?

Graph the region in the *xy*-plane.

a) Graph given *x* and *y* constraints.

b) AND find the xy-curves where the surfaces (the z's) intersect.

$z = 25 - x^2$

$z = 0$

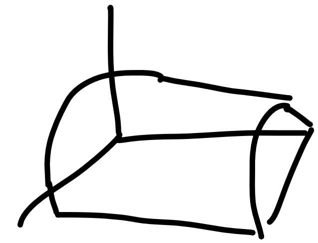
intersect @ $x = \pm 5$

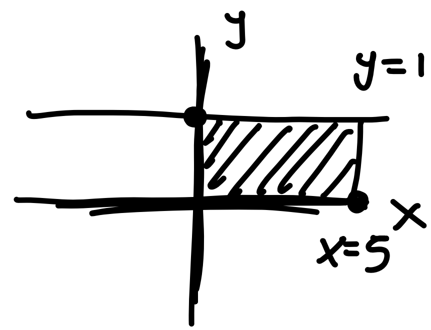
Example (directly from HW):

HW 15.1: Find the volume in the first octant ~~quadrant~~ bounded by z = 25 – x^2 and the plane y = 1.

$x=0$
$y=0$

① $\iint_R 25 - x^2 \, dA$

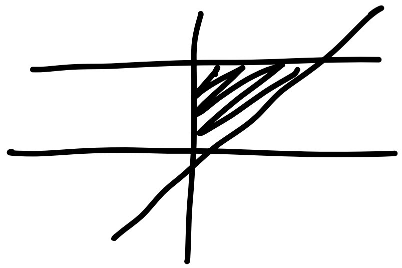


② 

$$\int_0^1 \left(\int_0^5 25 - x^2 \, dx \right) dy$$

HW 15.2: Find the volume enclosed by $z = 4x^2 + 4y^2$, $x = 0$, $y = 2$, $y = x$, and $z = 0$.

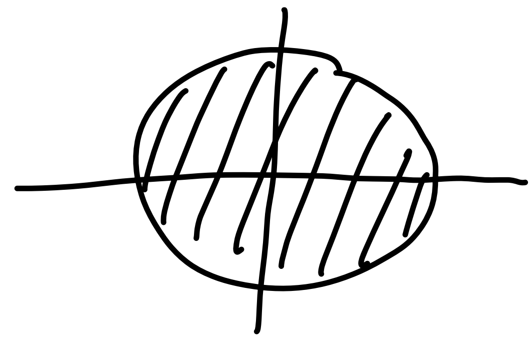
$$\iint_R 4x+4y^2 \, dA$$



HW 15.3: Find the volume below $z = 18 - 2x^2 - 2y^2$ and above xy-plane.

$z=0$

$$\iint_R 18-2x^2-2y^2 \, dA$$



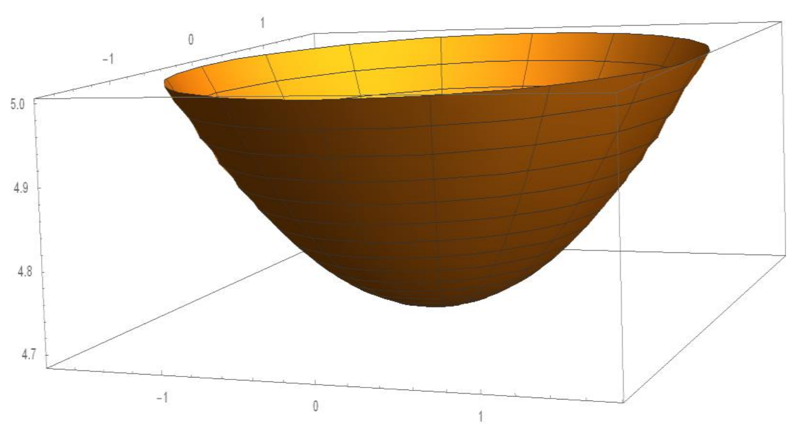
$$18-2x^2-2y^2 = 0$$

HW 15.3: Find the volume enclosed by $-x^2 - y^2 + z^2 = 22$ and $z = 5$.

HW 15.3: Find the volume above the upper cone $z = \sqrt{x^2 + y^2}$ and below $x^2 + y^2 + z^2 = 81$

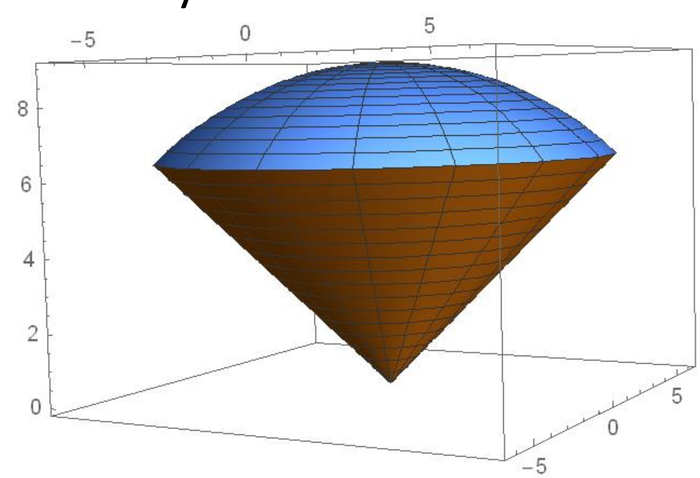
Volume enclosed by

$-x^2 - y^2 + z^2 = 22$ and $z = 5$.



The volume above the upper cone $z = \sqrt{x^2 + y^2}$ and below

$x^2 + y^2 + z^2 = 81$



Reversing the Order of Integration

1. Draw the region of integration for

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin(y)}{y} \, dy \, dx$$

then switch the order of integration.

2. Switch the order of integration for

$$\int_{0}^{4} \int_{\sqrt{x}}^{2} \sin(y^3)\, dy dx$$